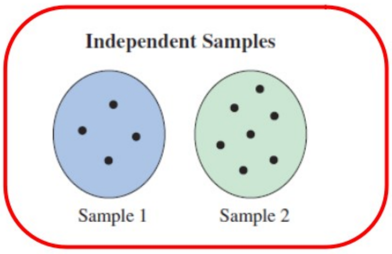
Testing the Difference Between Means - Small Samples

OBJECTIVES

- How to perform a two-sample *z*-test for the difference between two means μ_1 and μ_2 using small independent samples



In real life, it is often not practical to collect samples of size 30 or more from each of two populations. However, if both populations have a normal distribution, you can still test the difference between their means. A *t*-test is used to test the difference between two population means μ_1 and μ_2 using independent samples from each population. The following conditions are necessary to use a *t*-test for small independent samples.

1. The samples must be randomly selected.
2. The samples must be independent.
3. Each population must have a normal distribution

Use the *t*-test with

$$s_{\bar{x}_1 - \bar{x}_2} = \hat{\sigma} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

and d.f. $= n_1 + n_2 - 2$.

the population variances are equal

The standardized test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_{\bar{x}_1 - \bar{x}_2}}$$

Use the *t*-test with

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

and d.f. = smaller of $n_1 - 1$ and $n_2 - 1$.

the population variances are not equal.

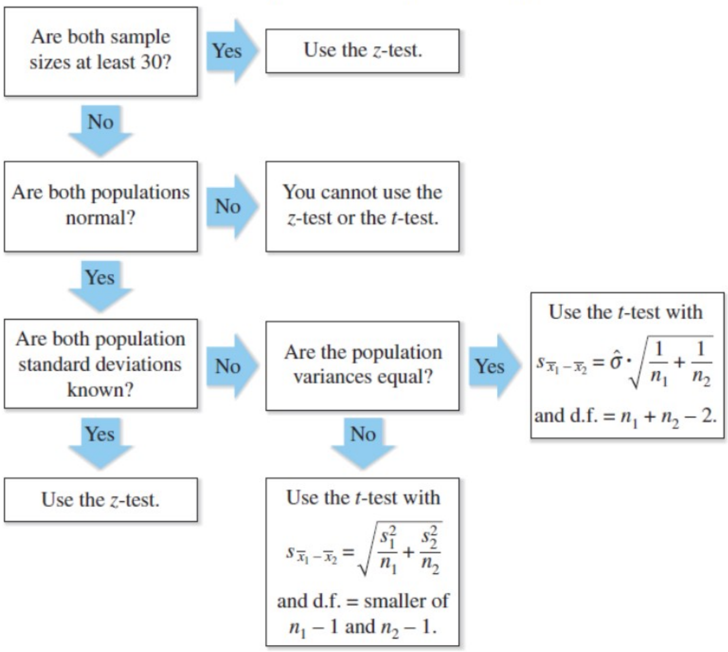
STUDY TIP

You will need to know whether the variances of two populations are equal. In this chapter, each example and exercise will state whether the variances are equal. You will learn to test for differences in variance of two populations in Chapter 10.

THE TWO-SAMPLE *t*-TEST FOR THE DIFFERENCE BETWEEN MEANS

The requirements for the z-test described in Section 8.1 and the t-test described in this section are shown in the flowchart below.

Two-Sample Tests for Independent Samples



THE TWO-SAMPLE t-TEST FOR THE DIFFERENCE BETWEEN MEANS

Using a Two-Sample t-Test for the Difference Between Means (Small Independent Samples)

IN WORDS	IN SYMBOLS
1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Determine the degrees of freedom.	d.f. $= n_1 + n_2 - 2$ or d.f. = smaller of $n_1 - 1$ and $n_2 - 1$
4. Determine the critical value(s).	Use Table 5 in Appendix B.
5. Determine the rejection region(s).	
6. Find the standardized test statistic and sketch the sampling distribution.	$t = \dfrac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)}{s_{\overline{x}_1 - \overline{x}_2}}$
7. Make a decision to reject or fail to reject the null hypothesis.	If t is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
8. Interpret the decision in the context of the original claim.	

Testing the Difference Between Means - Small Samples

EXAMPLE 1 A Two-Sample t-Test for the Difference Between Means

The results of a state mathematics test for random samples of students taught by two different teachers at the same school are shown at the left. Can you conclude that there is a difference in the mean mathematics test scores for the students of the two teachers? Use $\alpha = 0.10$. Assume the populations are normally distributed and the population variances are not equal.

Solution

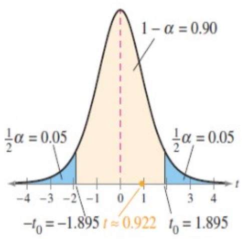
The claim is "there is a difference in the mean mathematics test scores for the students of the two teachers." So, the null and alternative hypotheses are

$H_0: \mu_1 = \mu_2$ and $H_a: \mu_1 \neq \mu_2$. (Claim)

Because the variances are not equal and the smaller sample size is 8, use d.f. = 8 − 1 = 7. Because the test is a two-tailed test with d.f. = 7 and $\alpha = 0.10$, the critical values are $-t_0 = -1.895$ and $t_0 = 1.895$. The rejection regions are $t < -1.895$ and $t > 1.895$. The standard error is

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$= \sqrt{\frac{39.7^2}{8} + \frac{24.5^2}{18}} \approx 15.1776.$$



The standardized test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_{\bar{x}_1-\bar{x}_2}}$$ Use the t-test.

$$\approx \frac{(473 - 459) - 0}{15.1776}$$ Assume $\mu_1 = \mu_2$, so $\mu_1 - \mu_2 = 0$.

$$\approx 0.922.$$

The graph at the left shows the location of the rejection regions and the standardized test statistic t. Because t is not in the rejection region, you should fail to reject the null hypothesis.

Interpretation There is not enough evidence at the 10% level of significance to support the claim that the mean mathematics test scores for the students of the two teachers are different.

Testing the Difference Between Means - Small Samples

▸ Try It Yourself 1

The annual earnings of 15 people with a high school diploma and 12 people with a bachelor's degree or higher are shown at the left. Can you conclude that there is a difference in the mean annual earnings based on level of education? Use $\alpha = 0.01$. Assume the populations are normally distributed and the population variances are not equal.

a. Identify the *claim* and state H_0 and H_a.
b. Identify the *level of significance* α and the *degrees of freedom*.
c. Find the *critical values* and identify the *rejection regions*.
d. Find the *standardized test statistic t*. *Sketch* a graph.
e. *Decide* whether to reject the null hypothesis.
f. *Interpret* the decision in the context of the original claim.

Sample Statistics for Annual Earnings

High school diploma	Bachelor's degree or higher
$\bar{x}_1 = \$27{,}136$	$\bar{x}_2 = \$34{,}329$
$s_1 = \$2318$	$s_2 = \$4962$
$n_1 = 15$	$n_2 = 12$

Testing the Difference Between Means - Small Samples

EXAMPLE 2 A Two-Sample t-Test for the Difference Between Means

A manufacturer claims that the mean calling range (in feet) of its 2.4-GHz cordless telephones is greater than that of its leading competitor. You perform a study using 14 randomly selected phones from the manufacturer and 16 randomly selected similar phones from its competitor. The results are shown at the left. At $\alpha = 0.05$, can you support the manufacturer's claim? Assume the populations are normally distributed and the population variances are equal.

Sample Statistics for Calling Range

Manufacturer	Competitor
$\bar{x}_1 = 1275$ ft	$\bar{x}_2 = 1250$ ft
$s_1 = 45$ ft	$s_2 = 30$ ft
$n_1 = 14$	$n_2 = 16$

Solution

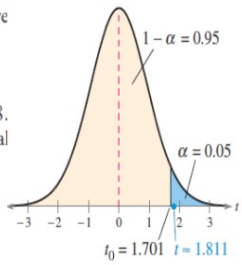
The claim is "the mean calling range of the manufacturer's cordless phones is greater than that of its leading competitor." So, the null and alternative hypotheses are

H_0: $\mu_1 \leq \mu_2$ and H_a: $\mu_1 > \mu_2$. (Claim)

Because the variances are equal, d.f. $= n_1 + n_2 - 2 = 14 + 16 - 2 = 28$. Because the test is a right-tailed test with d.f. $= 28$ and $\alpha = 0.05$, the critical value is $t_0 = 1.701$. The rejection region is $t > 1.701$. The standard error is

$$s_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= \sqrt{\frac{(13)(45^2) + (15)(30^2)}{14+16-2}} \cdot \sqrt{\frac{1}{14} + \frac{1}{16}} \approx 13.8018.$$



The standardized test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_{\bar{x}_1-\bar{x}_2}}$$ Use the t-test.

$$\approx \frac{(1275 - 1250) - 0}{13.8018}$$ Assume $\mu_1 = \mu_2$, so $\mu_1 - \mu_2 = 0$.

$$\approx 1.811.$$

The graph at the left shows the location of the rejection region and the standardized test statistic t. Because t is in the rejection region, you should decide to reject the null hypothesis.

Interpretation There is enough evidence at the 5% level of significance to support the manufacturer's claim that its phones have a greater calling range than its competitor's.

Testing the Difference Between Means - Small Samples

Try It Yourself 2

A manufacturer claims that the watt usage of its 17-inch flat panel monitors is less than that of its leading competitor. You perform a study and obtain the results shown at the left. At $\alpha = 0.10$, is there enough evidence to support the manufacturer's claim? Assume the populations are normally distributed and the population variances are equal.

a. Identify the *claim* and state H_0 and H_a.
b. Identify the *level of significance* α and the *degrees of freedom*.
c. Find the *critical value* and identify the *rejection region*.
d. Find the *standardized test statistic t*. *Sketch* a graph.
e. *Decide* whether to reject the null hypothesis.
f. *Interpret* the decision in the context of the original claim.

Sample Statistics for Watt Usage

Manufacturer	Competitor
$\bar{x}_1 = 32$	$\bar{x}_2 = 35$
$s_1 = 2.1$	$s_2 = 1.8$
$n_1 = 12$	$n_2 = 15$

Testing the Difference Between Means - Small Samples

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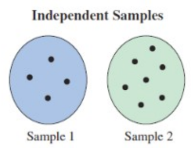

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Use the *t*-test with

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